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Experiments in integrating the vorticity advection equation by a Lagrangian method.

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1. Introduction

The basic equation used in numerical forecasts of 500 mb. height with the barotropic model is

$$\nabla^2\left(\frac{\partial\psi}{\partial t}\right) = J(\nabla^2\psi + f, \psi) \quad - - - (1)$$

where ψ is a stream function such that $u = -\partial\psi/\partial y$, $v = \partial\psi/\partial x$ and f is the Coriolis parameter.

Numerical calculations based on the similar equation

$$\nabla^2\left(\frac{\partial\psi}{\partial t}\right) = J(\nabla^2\psi, \psi) \quad - - - (2)$$

applied to fields of ψ represented by analytical functions have demonstrated that the finite difference approximations commonly used in numerical forecasting lead to significant truncation errors particularly when the dimensions of the circulation systems concerned are only a few grid lengths.

There is therefore some reason to seek methods of solution of equation (1) which are less subject to truncation error. The present note reports the result of some numerical studies of the properties of such an alternative integration system. The investigations were carried out using equation (2) because some exact solutions were known for specified analytical forms of ψ, but the method is applicable by slight extension to equation (1).

Equation (2) may be rewritten

$$\frac{D}{Dt}(\nabla^2\psi) = 0$$

representing the conservation of vorticity, $\nabla^2\psi$, following the motion of the fluid. The basis of the present method was to commence the time-integration with a cloud of fluid elements, distributed over the area of integration and to attribute to each the initial vorticity of the fluid at its initial location. Subsequently the position of each 'fluid element' was calculated from its displacement with the velocity of the fluid and at suitable intervals the field of the streams function ψ was reconstructed from the distribution of vorticity, $\nabla^2\psi$, as given by the fluid elements each retaining its initial vorticity. The new ψ field provided a new wind field, further displacements of the fluid elements could be calculated and the integration continued step by step.

Additional advantages of the technique are that the vorticity of the fluid is automatically conserved and fictitious changes cannot be introduced by truncation error (although the vorticity may be wrongly located). Moreover, the limitation on the length of the time step in the integraion arising from the Courant, - Friedrichs - Lewy stability criterion does not apply and longer time-steps are possible than with the simple finite difference procedure.

The use of Lagrangian methods in solving both barotropic and baroclinic forecast equations has already been advocated by Wiin - Nielsen (1959). He recommends two procedures both of which differ from that discussed in the following paragraphs. His first method considers the motion of a single set of fluid elements only over a period during which they can be advected in a space-smoothed stream field which can be considered as unchanging over the period. At the end of this period (say 12 hours in a practical case) a new set of fluid elements is formed and followed. This method might lose some of the advantage of strict vorticity conservation. Wiin-Nielsen's second method works strictly in Lagrangian

coordinates and leads to more complicated calculations. These considerations influenced the choice of method which has been studied below.

Experimental integrations reported in the present note have all been carried out on stream functions represented by artificial analytical fields and the results compared with those obtained by usual finite difference methods and with the known correct solution.

2. The integration method in outline.

The following notation will be used:-

u , v components of velocity in x and y directions
ψ stream function such that

$$u = -\frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x} \qquad - \quad - \quad (4)$$

$X_{r,t}$, $Y_{r,t}$ coordinates of rth fluid element at time t

Although the method is Lagrangian a fixed square grid is employed and the symbol ∇ will be used to denote the simple finite difference approximation to the operator ∇^2 . Thus

$$\{\nabla \psi\}_0 = \frac{1}{h^2}\{\psi_1 + \psi_2 + \psi_3 + \psi_4 - 4\psi_0\}$$

where h is the grid length and ψ_0, ψ_1 etc. are the values of ψ at an array of points as illustrated in Fig.1.

Z_r is defined as the finite difference approximation to the vorticity of the rth fluid element at its initial position.

T is time interval between time steps.

The integration procedure can be divided into 8 steps which will be described more fully below. The steps are as follows:-

1. Form a list of fluid elements and associated vorticity Z_r calculated from initial stream function .
2. Calculate the advecting field of u' and v' from ψ
3. Interpolate u' and v' to positions of the fluid elements.
4. Calculate displacement of fluid elements and new positions.after one time step.
5. Revise the list of fluid elements to take account of those leaving or entering the integration area.
6. Form the field of vorticity, Z, at grid points.
7. Calculate a new field of ψ from Z
8. Repeat from step 2.

In more detail the procedure adopted in individual steps was as below.

Step 1

The initial list of fluid elements contained one entry for each grid point in the basic grid including boundary points. With each fluid element was associated its initial vorticity which remained unaltered during the calculation. This was calculated from the simple finite difference approximation

$$Z_{r,0} = \{\nabla^2 \psi\}_{t=0} \qquad - \quad - \quad - \quad (5)$$

Step 2

Fjørtoft (1955) has demonstrated from equation (3) that

$$\frac{D'}{Dt}(\nabla^2\psi) = 0$$

where D'/Dt denotes differentiation following a fictitious motion determined by a stream function ψ'

where $$\psi' = \psi + a\nabla^2\psi$$

The field of ψ' can be made smoother than that of ψ by suitable choice of a and it then changes more slowly with time. These properties make ψ' more suitable as an advecting field than ψ leading to smaller truncation errors and the possibility of longer time steps. Wiin-Nielsen (1959) recommends the use of $a = h^2/4$ and this has been adopted in the present experiments.

Step 2 therefore consisted of calculating the smoothed stream function from the relation (with notation of Fig.1)

$$\psi_0' = \frac{1}{4}(\psi_1 + \psi_2 + \psi_3 + \psi_4)$$

Subsequently the advecting velocity components u' and v' were calculated at all grid points including boundary points. Simple 2-power finite differences were used (one-sided differences being used when necessary at boundary points)

At interior points

$$u_0' = -\frac{1}{2h}(\psi_2' - \psi_4')$$

$$v_0' = \frac{1}{2h}(\psi_1' - \psi_3')$$

Step 3

After the initial time step the fluid elements are found, not at grid points, but within the squares which form the mesh. Interpolation to the position $P(r, s)$ of the fluid element was carried out by means of the formula,

$$f_p = (1-s)\{(1-r)f_0 + rf_1\} + s\{(1-r)f_2 + rf_3\} \quad -- \quad (7)$$

Suffixes refer to positions in Fig.2 and f may represent either u or v .

Step 4

At each time step except the first the value of the coordinates of the fluid element for use at the next time step were calculated from the centred difference relations

$$X'_{t+1} = X_{t-1} + 2u'_t\tau$$

$$Y'_{t+1} = Y_{t-1} + 2v'_t\tau$$

where X' and Y' are preliminary values for use at the next time step.

The value of X'_t was however also corrected to give a final value where

$$\left.\begin{aligned} X_t &= X_{t-1} + \tfrac{1}{2}(u'_t + u'_{t-1})\tau \\ Y_t &= Y_{t-1} + \tfrac{1}{2}(v'_t + v'_{t-1})\tau \end{aligned}\right\} \quad (9)$$

At the first time step uncentred differences were used and in place of (8) the relations were

$$\left.\begin{aligned} X'_1 &= X_0 + u'_0\tau \\ Y'_1 &= Y_0 + v'_0\tau \end{aligned}\right\} \quad (10)$$

Step 5.

All fluid elements with new coordinates outside the boundary of the grid were omitted from the list. The list was then brought up to its original length by adding new fluid elements positioned at boundary points. For this purpose priority was given to boundary points distant more than one grid length from any element already in the list.

For the purpose of testing the integration procedure two techniques were used - one assumed that true boundary conditions were known and the other that boundary values were fixed. In the former the new points were introduced with correct vorticity values. In the latter the vorticity attributed to the new fluid element was that initially present at the boundary point.

Step 6.

The field of vorticity at the grid points was reconstructed from the vorticity values of the displaced fluid elements by sharing the vorticity carried by the fluid element between the grid points which formed the corners of the square in which it lay. Thus in figure 2 a fluid element vorticity Z_r at point P would make a contribution to the vorticity at grid points 0, 1, 5 and 2 of $(1-r)(1-s)Z_r$, $r(1-s)Z_r$, $rs\,Z_r$ and $(1-r)s\,Z_r$ respectively.

Step 7

The new field of ψ_t was calculated from the field of vorticity from the equation

$$\nabla^2 \psi_t = Z_t \qquad (12)$$

which was solved by the usual iterative Liebmann process. The values of ψ_{t-1} were used as a first guess to ψ_t and over-relaxation by a factor of 4/3 was employed. The true boundary values of ψ_t were used in most experiments but the effect of fixed boundary value for ψ_t was also examined.

3. Some details of the computing programme.

In order to economise computer storage and transfers to and from the magnetic store, all information in the list of fluid elements was stored as 10-bit numbers. For each fluid element the information consisted of

$X_{r,t}$, $Y_{r,t}$, Z_r, N, $X_{r,t-1}$, $Y_{r,t-1}$, u'_{t-1} and v'_{t-1}.

Values of X and Y were rounded off to the nearest 1/32 of a grid length. N is the number of the square in which the point lies and is convenient for locating the appropriate values of functions at grid-points.

Calculations were carried out on METEOR for a 19 x 19 grid of points. They took from 1 to 2 minutes per time step depending upon the number of iterations carried out in solving the Poisson equation in Step 7. This is discussed further below.

4. Comparison with Eulerian finite difference integrations.

Integrations of the barotropic advection equation (2) were carried out by the Lagrangian method on the analytical field of ψ given by

$$\psi = 2 - \frac{1}{1 + \frac{x^2 + y^2}{a^2}} - \frac{y}{b}$$

which represents a circular vortex superimposed upon a uniform current. By adjustment of the constants a and b the size and the intensity of the vortex relative to the stream could be varied. The vortex should move along the x-axis without change of form if equation (2) is satisfied. In all calculation the grid-length h is taken as unity. If the grid-length is taken to correspond to that used currently in numerical forecasting (C.160 mm) then a = 4 gives a trough comparable in dimension with a large 500 mb. trough, a = 2 represents a small system such as an active depression and a = 1 a very small but not impossible - synoptic system.

The results of an integration in 8 steps with a = 4, b = 8 are illustrated in Figure 3(a), and (b). Figure 3(c) gives the corresponding result using conventional finite difference methods using 16-steps. The time-interval was chosen so that the vortex should have moved four grid lengths to the right in the absence of truncation error. (A minimum of 10 time steps are required by the Courant-Friedrichs-Lewy stability criterion). Boundary values were assumed known throughout the integrations.

The errors of the Lagrangian integration are acceptably small - less than 3% of the maximum changes. Moreover the errors grow very slowly with time; the first uncentred time step introducing errors comparable with the final error (see Fig.4).

The errors of the integration carried out by the conventional two-point finite differences are however substantially larger. Moreover the finite difference integration systematically under-estimates the displacement of the vortex, whereas the Lagrangian integration fixes the centre of vortex within a small fraction of a grid length.

Experiments have demonstrated that the errors of the usual finite difference methods increase rapidly as the size of the vortex is reduced. The accuracy of the Lagrangian integration method does not deteriorate in the same way. This is strikingly demonstrated by the results of integrations in which the constants in equation (13) are chosen as a = 1, b = 8. Initial field and results are shown in Figs.5(a), (b) and (c). The conventional finite difference integration Fig.5(c) has reduced the intensity of the vortex by half, moved it at 20% less than the true speed and considerably distorted the shape. Although the Courant Friedrichs-Lewy stability criterion has been exceeded over a limited area results cannot be improved by shorter time-steps. None of these defects is present in the integration carried out by the Lagrangian technique. The errors are only a little greater than with the larger vortex a = 4 - the maximum error is less than 5% of the maximum change.

A substantial improvement over the conventional finite difference procedure is also found for a = 2.

If 5-point differences are used in the Eulerian procedure errors may be reduced for the larger vortices (a = 4) and results are then better than by the Lagrangian method. However for the smaller vortices (a = 2) and (a = 1) the improvement possible by means of the 5-point formula is slight and the Lagrangian technique gives substantially better results.

5. Effect of change in length of time step.

The integration illustrated in Fig.3 (a = 4 b = 8) has been carried out in 16, 8, 4 and 2 time steps.

The errors of the final ψ -fields were as follows:-

Table I - Errors of Lagrangian integration for varying length of time step.

Number of time steps	16	8	4	2
Largest error (without regard to sign)	0·021	0·017	0·036	0·079
Mean of 4 larges errors (without regard to sign)	0·020	0·016	0·019	0·079

As the time step was lengthened there was a systematic tendency to "fill" the vortex but no significant error in position.

These results suggest that results of acceptable accuracy can be obtained up to a time step of one quarter of the total integration period in this case. During such a time step the vortex moves one grid-length and some individual fluid elements move more than two grid lengths. A corresponding time step in a practical barotropic forecast would be 4 to 6 hours.

6. Higher order finite-differences

Experiments were conducted to determine whether the results of the Lagrangian integrations would be improved if higher order finite differences were used at Step 2 to calculate ω_c and if non-linear interpolation were used in step 3. However, neither of these changes produced any improvement in the result of the integration.

7. Effects of smoothing at Step 2.

In order to determine the desirability of including space-smoothing of the field at step 2, integrations have been carried out both with the original field ψ as the advecting stream function and with the space-smoothed field ψ'. Very little difference in the results occurs for a = 4, but for smaller vortex diameter, especially for a = 1, the smoothing leads to substantial improvements. The size of errors with and without smoothing is compared in Table II. (8 time steps were used in the integration).

Table II - Comparison of errors with and without space-smoothing of the field

(a) without smoothing (b) with smoothing.

a	4		2		1	
	(a)	(b)	(a)	(b)	(a)	(b)
Largest error (without regard to sign)	0·017	0·017	0·111	0·127	0·109	0·030
Mean of 4 largest errors (without regard to sign)	0·016	0·016	0·063	0·039	0·081	0·027

8. Number of iterations required to solve the Poisson equation.

Experiments showed that some 60 to 80 iterations were need to solve the Poisson equation (12) with adequate accuracy, and in order to achieve this the iteration procedure was continued until both of the following criteria were criteria were satisfied

(i) Largest change in computed value of $\psi \leq 2^{-15}$

(ii) Algebraic sum of $\nabla\psi_r - Z_r$ over all grid points lies between -2^{-9} and $+2^{-9}$

The value of ψ at the previous time step was used as an initial guess from which to start iterations.

As a large part of the computing time was taken in solving the Poisson equation the possibility of reducing the number of iterations was examined.

During the latter part of the convergence it was noted that the remaining error consisted of more or less uniform errors over much of the field. Gradients of ψ were not affected by large errors. It was therefore thought that the number of iterations could be reduced at intermediate time steps because the resulting ψ fields were used only to evaluate gradients of ψ. This proved to be the case.

Two series of integrations were carried out; one using 10 iterations at all time steps except the last, the other using 20 iterations at all intermediate time steps. In both cases iteration was continued at the final time step until both criteria (i) and (ii) above were satisfied.

The integrations in which 20 iterations were made did not show any consistently greater errors than those in which iterations were continued to 60 or more in order to satisfy criteria (i) and (ii). Similarly the results of computations for the smaller vortices (a = 2 and a = 1) were not worsened when iterations were reduced to 10, but the computations for the larger vortex (a = 4) showed a systematic tendency to displace the vortex to the left of its true path. This arises because iteration starts from ψ_{t-1} which, regarded as a field of ψ, has positive errors ahead of the vortex and negative errors behind. If iteration is not carried to completion an error field will remain which produces an excessive gradient of ψ_t in the x-direction. This will introduce a fictitious component of the vortex motion to the left of its true track.

These experiments indicate that iterations may be reduced to about 15 per time step. They might be reduced further if systematic errors could be eliminated from the "initial guess" used to start iterations. One practical method would be to extrapolate from ψ_{t-2} and ψ_{t-1}.

9. Boundary conditions.

One integration was carried out by the Lagrangian procedure in which the boundary values of ψ and ζ were held constant throughout the integration. This is similar to the procedure which would have to be adopted in the practical forecasting problem in which no information was used outside the grid area. The results are shown in Fig.6, together with the results of a similar integration using the usual finite difference procedure and ψ constant on boundary of both 19 x 19 and 17 x 17 grids. The Lagrangian integration was carried out in 4 time steps and the Eulerian in 16. The initial field of ψ was that shown in Fig.3(a) and the vortex should have been displaced four grid lengths to the right without change of form.

Both integrations show large systematic errors of the same general character. These arise from the erroneous boundary conditions assumed. The errors are somewhat greater in the Eulerian method because it is necessary to apply the assumption of no change at the first inner row of the grid as well as the boundary. The former is nearer to the vortex and errors are necessarily larger on it. The error field in the Lagrangian results is also smoother and shows no sign of the computational irregularity appearing near the right-hand boundary in the Eulerian results.

It can be deduced from the foregoing results that erroneous boundary conditions will introduce no more error into integrations by the Lagrangian method than into those by the more conventional Eulerian technique.

10. Other variants of the Lagrangian techniques.

Two other variants of the Lagrangian technique were explored but without advantage.

Firstly, the number of 'fluid elements' employed as a basis for the integration was quadrupled. Although some minor errors remained in the programme in respect of the treatment of the boundaries, it was clear that no advantage, was gained when the number of 'fluid elements' was increased, but computation time was materially greater.

Second, using the denser field of 'fluid elements' (four times as many as grid points) an alternative procedure was used to determine the vorticity at grid points in 'Step 6' from the vorticity of the 'fluid elements'. In this procedure the vorticity at the grid point, O, was taken to be the average vorticity of all 'fluid elements' found within the shaded square in Fig.3.

This procedure proved unsatisfactory because the 'weight' given to the vorticity of any 'fluid element' depended on the number of other 'fluid elements' in the same square. As a result the total vorticity attributed to the whole fluid fluctuated irregularly with consequent irregular variations of the stream function.

11 Discussion and recommendations.

The Lagrangian technique of integration of the vorticity advection equation which has been explored in the present paper has two important advantages over the Eulerian finite difference procedure generally employed hitherto in numerical forecasting. These are

(a) The Lagrangian method treats with adequate accuracy disturbances of the flow with dimensions of only a few grid lengths. (The Eulerian system introduces very substantial errors in respect of these systems).

(b) The length of the time-step may be considerably greater in the Lagrangian system than in the Eurlerian technique. The Lagrangian integration procedure is not subject to computational instability of the Courant-Friedrichs-Lewy type and satisfactory results are achieved when disturbances of the flow move as much as one grid length in a single time step.

There are also (c) indications that boundary effects may be somewhat less troublesome in the Lagrangian technique.

As applied to numerical forecasts with the barotropic model it may reasonably be expected that grid-lengths may be increased to 200 nm or 250 nm and time steps to 6 hrs. with an accuracy as good or better than current Eulerian techniques. The technique is particularly suited to hemispheric forecasts for extended periods.

The computation time for each time step is somewhat greater for the Lagrangian technique than for the Eulerian but this is more than compensated by the reduced number of time steps. An overall reduction of integration time by 30 to 50% might reasonably be achieved.

Further improvement of the Lagrangian technique might be obtained by

(a) repeating the first time step so as to reduce the errors introduced by the first uncentred differences.

(b) extrapolating the ψ- field linearly to provide a first guess for iterative solution of the Poisson equation at 'Step 1'. Thereby the number of iterations might be reduced and the overall process 'speeded up'.

The next step would appear to be the rewriting of the computation programme to carry out barotropic forecasts on the standard 20 x 24 grid for comparison with the Eulerian results. The programme should contain facilities for testing the refinements (a) and (b) above. Experiments in applying the Lagrangian technique to hemispherical forecasts would also be desirable.

26th July, 1960.

References

Wiin-Nielsen, A. 1959 — On the application of trajectory methods in numerical forecasting. Tellus Vol.11 p.180

Fjørtoft, R. 1955 — On the use of space-smoothing in physical weather forecasting Tellus Vol.7 p.462

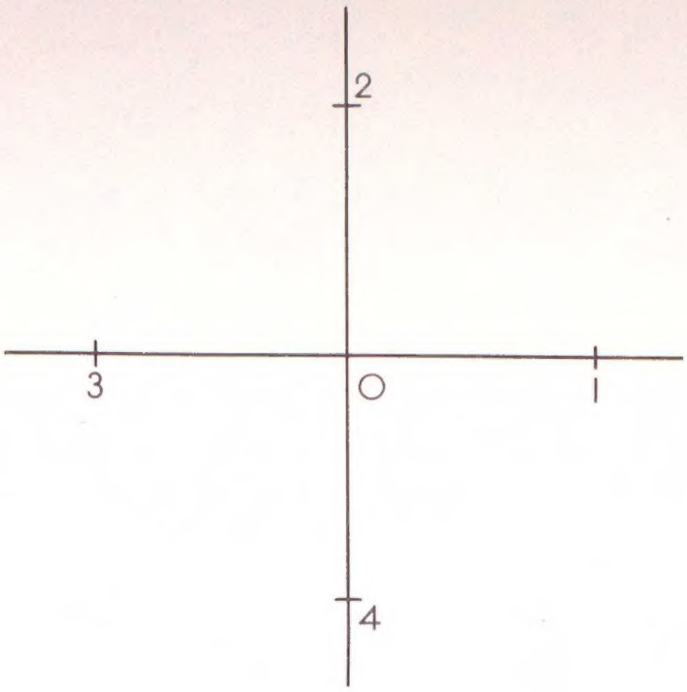

Fig. 1

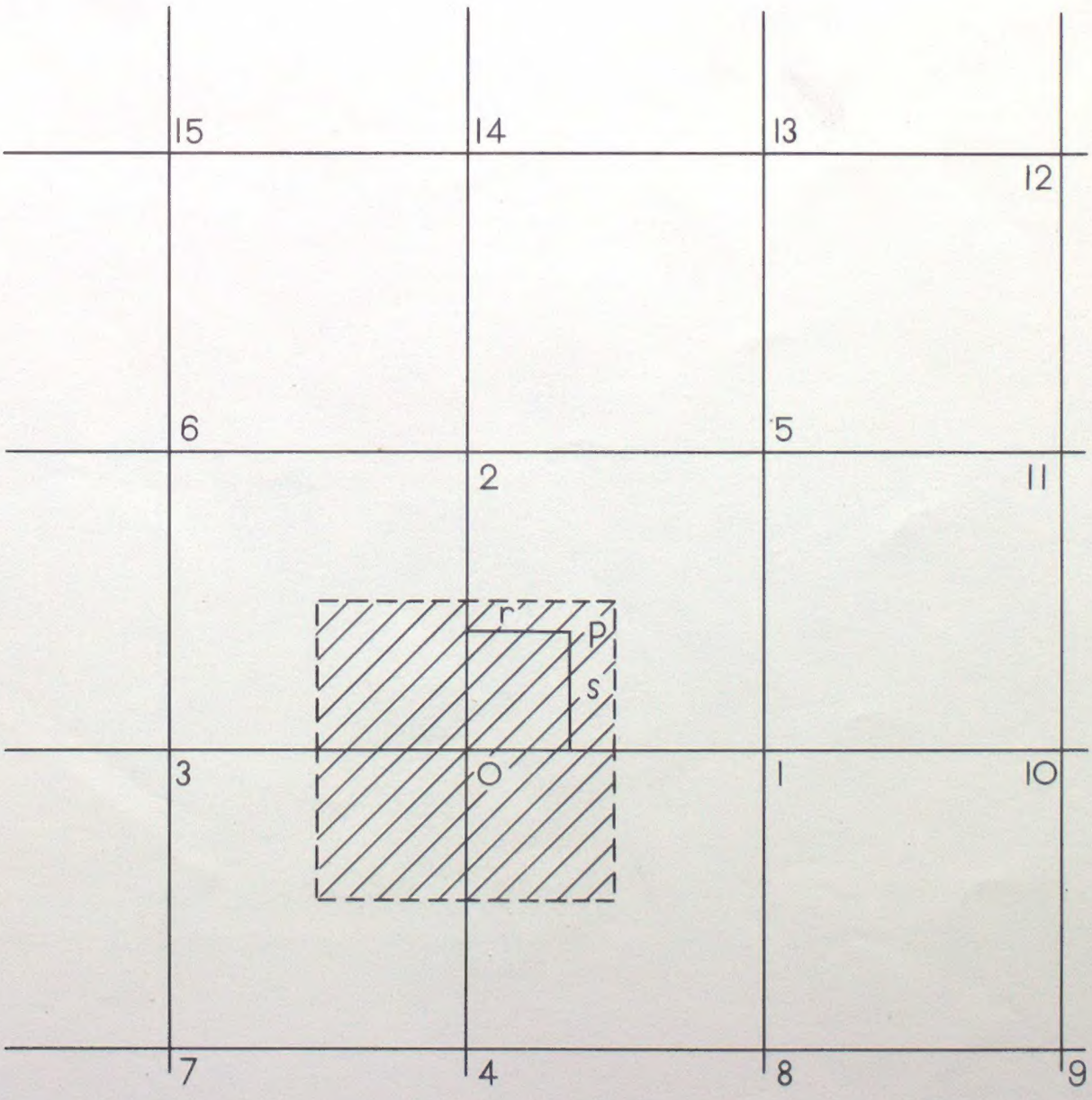

Fig. 2

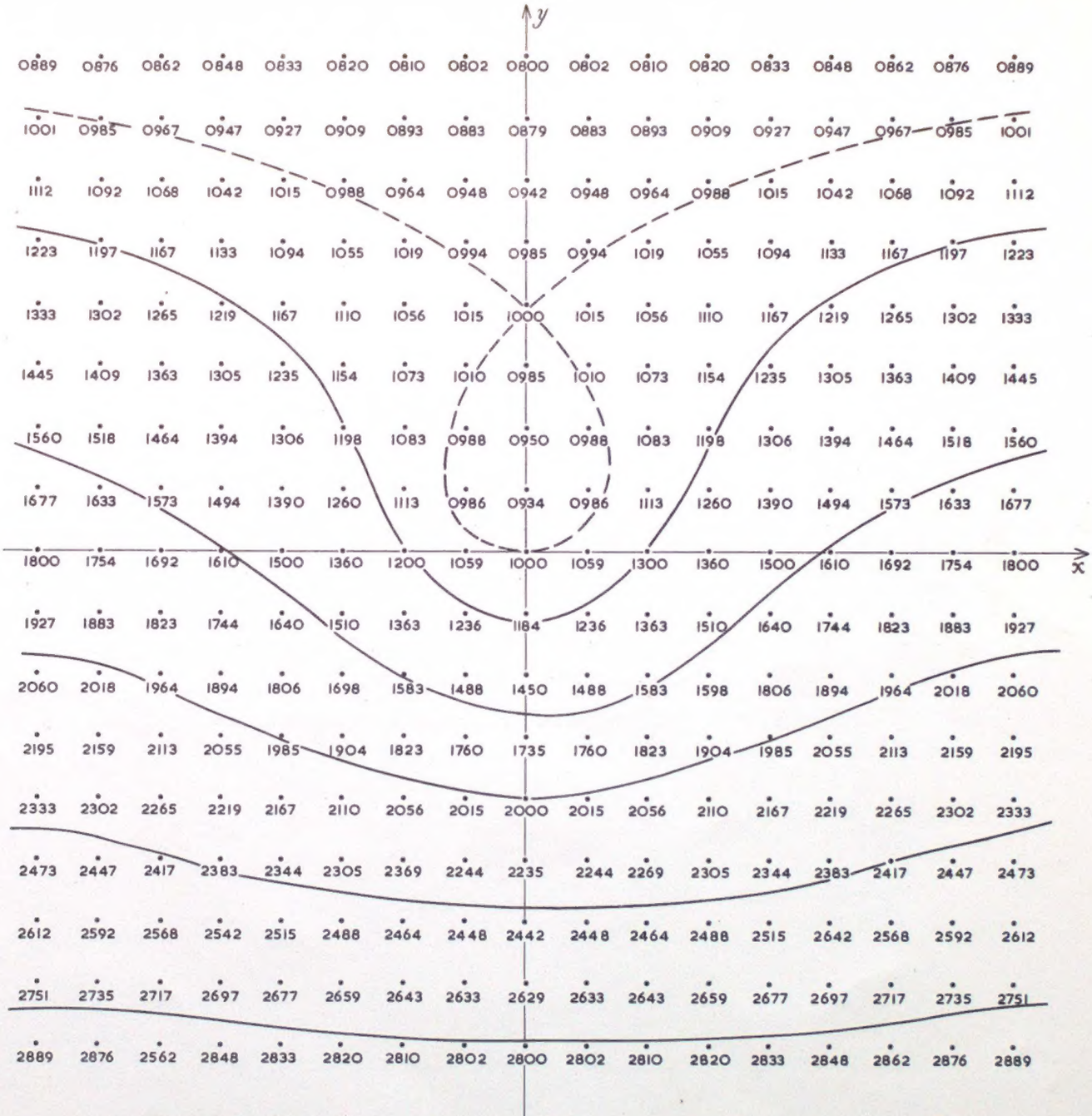

Fig.3(a) Initial field ψ a=4 b=8 All figures are multiplied by 10^3

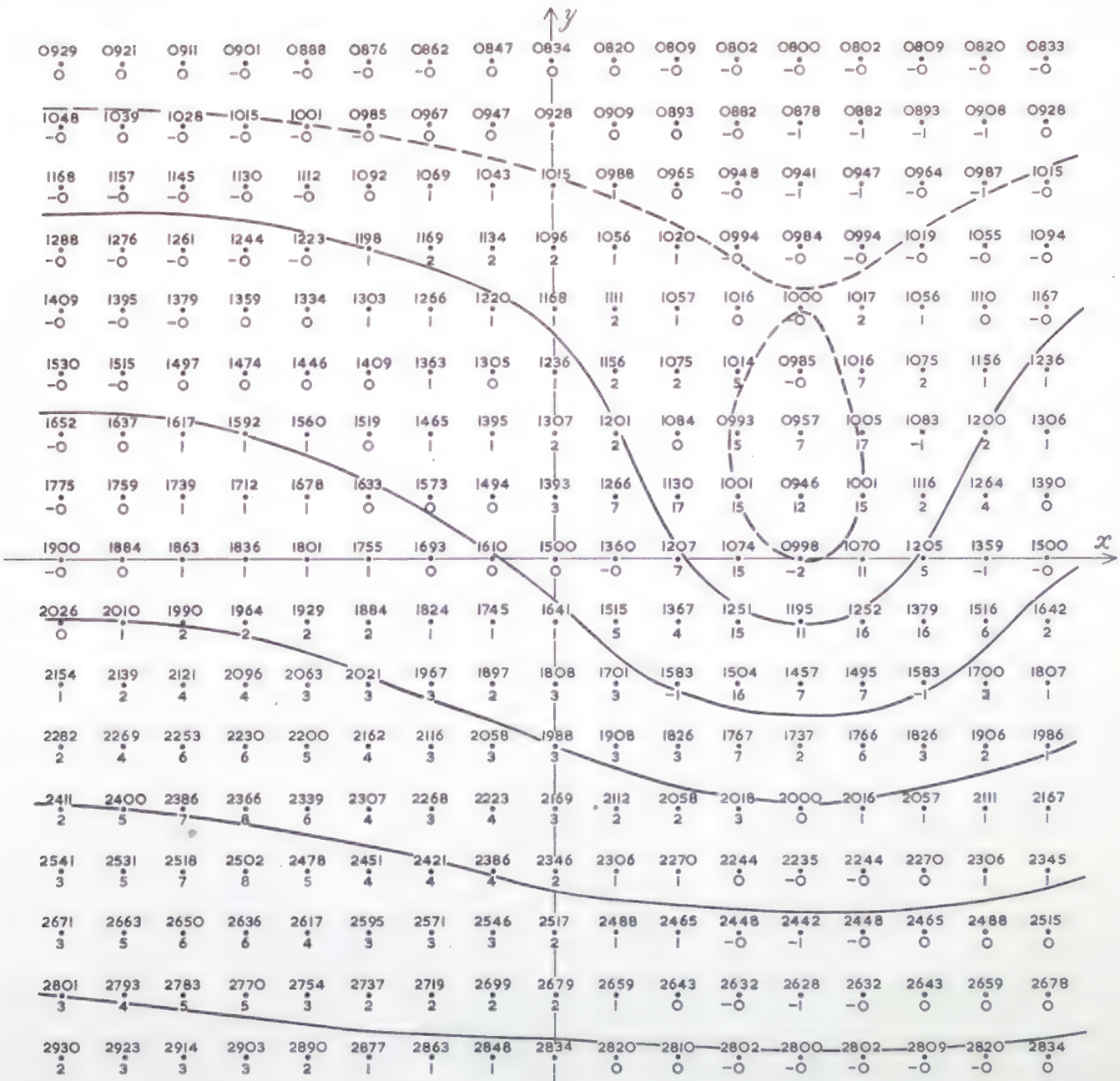

Fig.3 (b) Final field of ψ and error field (Lagrangran Method) a = 4 b = 8 (8 time steps)

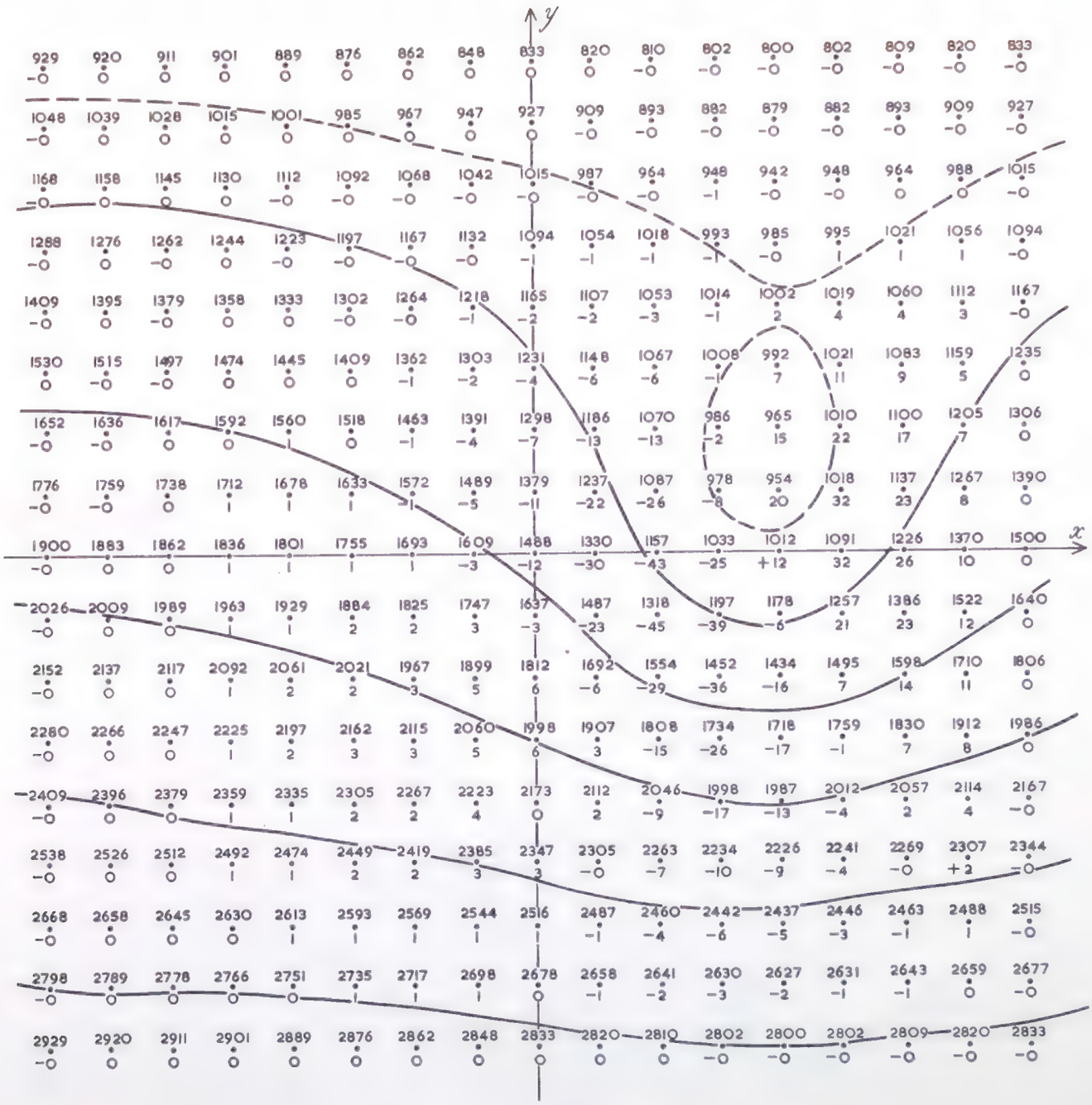

Fig. 3 (c) Final field of ψ and error field (Eulerian method)
a=4 b=8 16 steps. All figs multiplied by 10^3

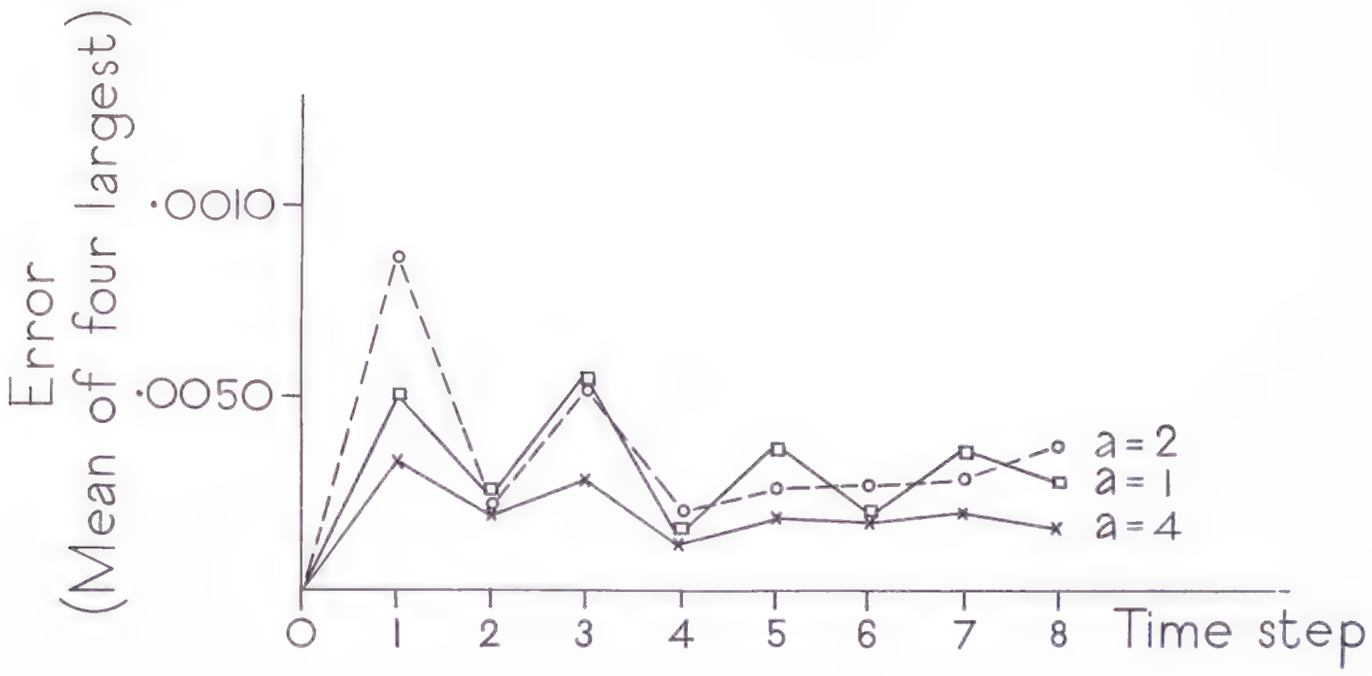

Fig. 4 Growth of error with time step
(Lagrangran Method)

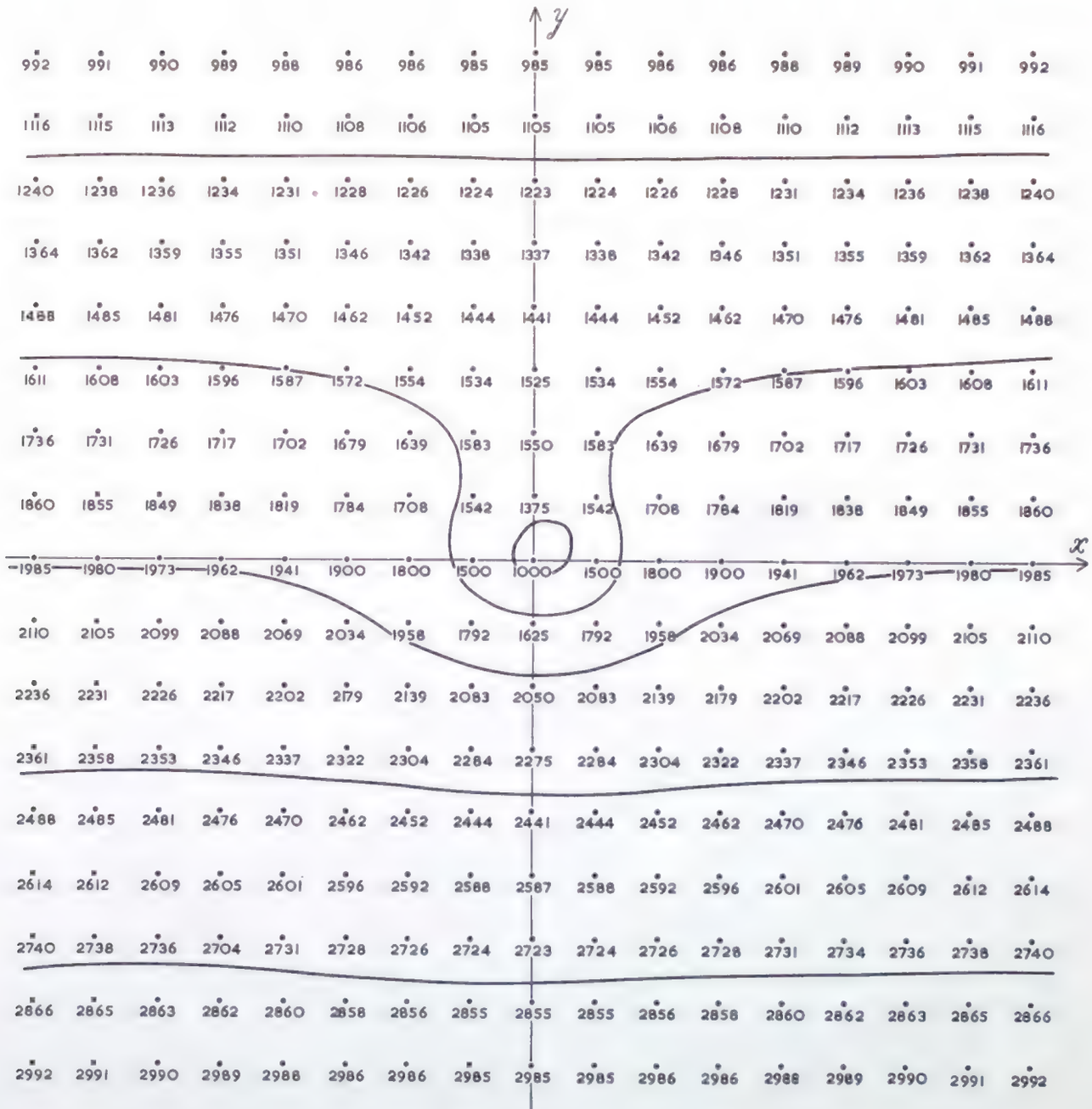

Fig. 5(a) Initial field ψ a=1 b=8 All figures multiplied by 10^3

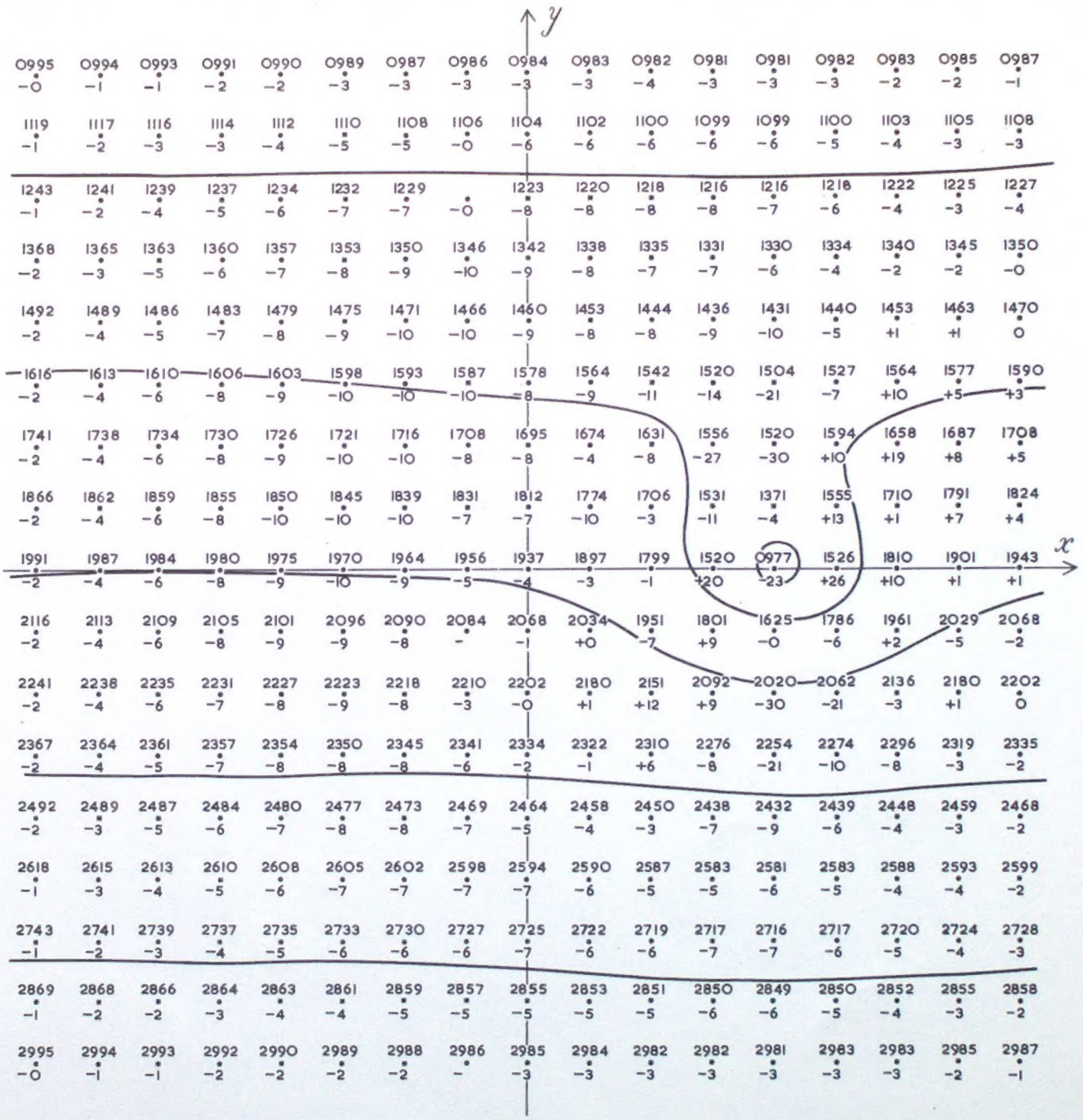

Fig.5(b) Final field of ψ and error field (Lagrangran Method) a=1 b=8 (8 time steps)

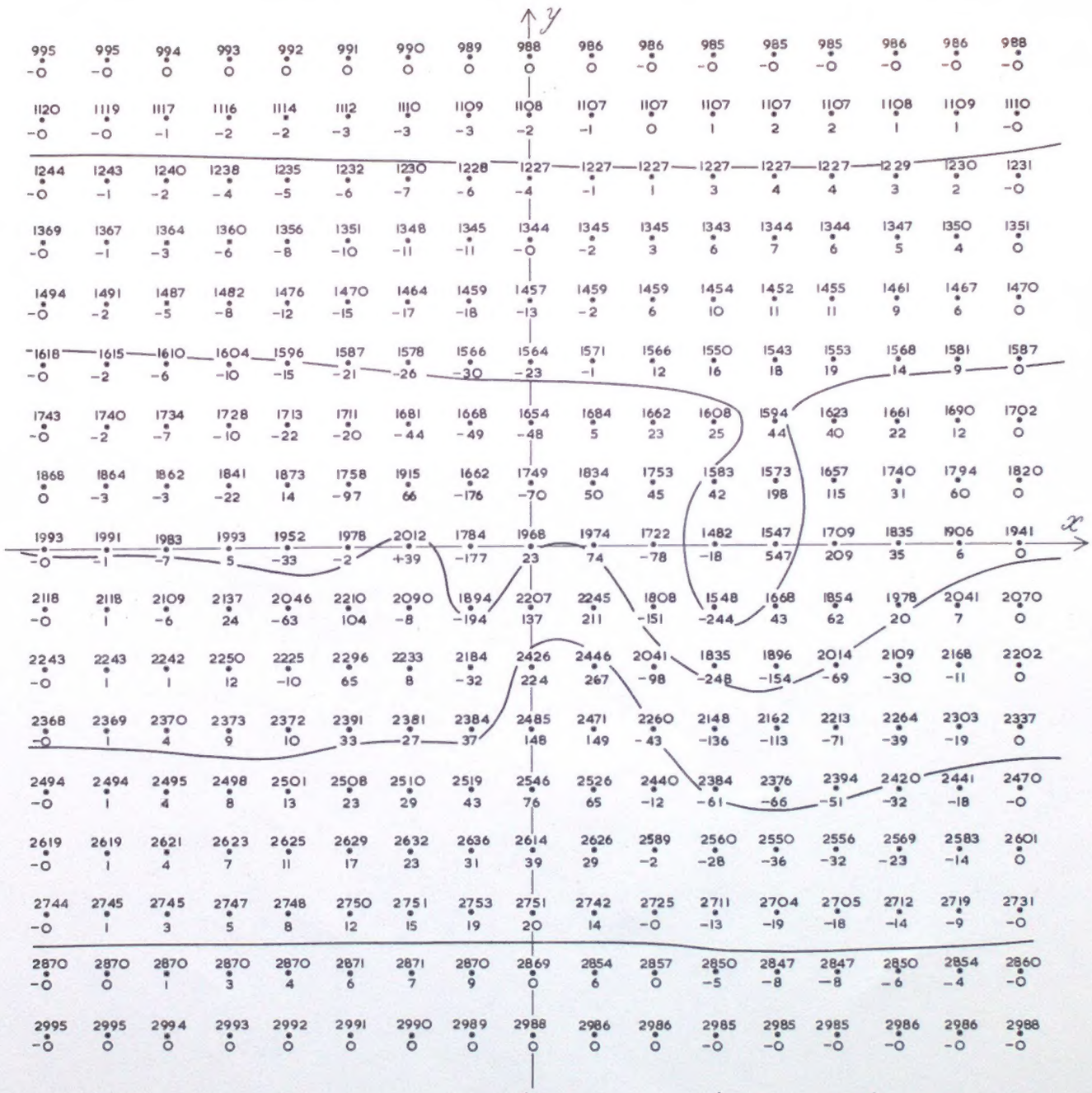

Fig. 5(c) Final field of ψ and error field (Eulerian method) a=1 b=8 (16 time steps)

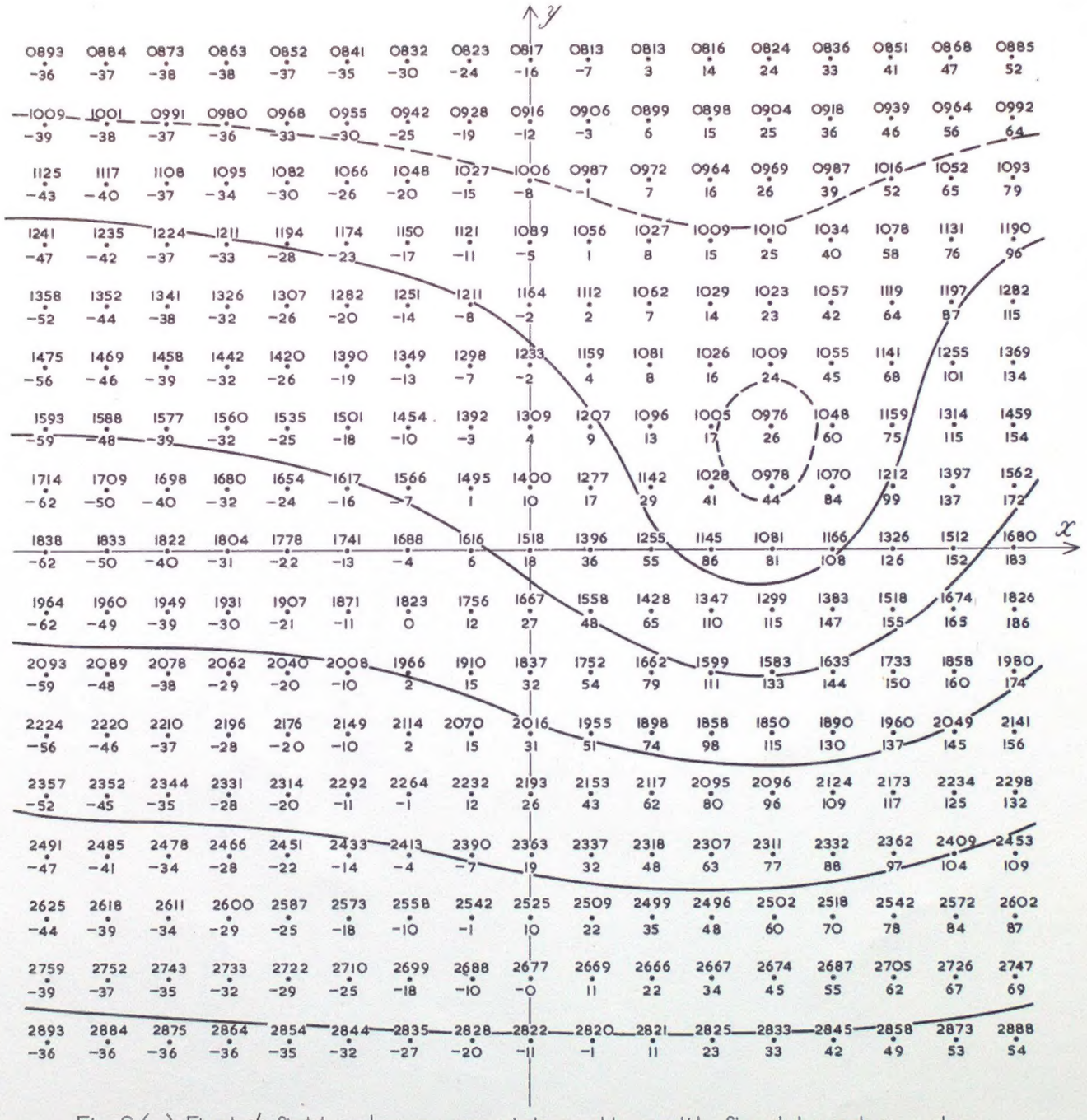

Fig. 6 (a) Final ψ field — Lagrangran integration with fixed boundary values

a = 4 b = 8 (4 time steps)

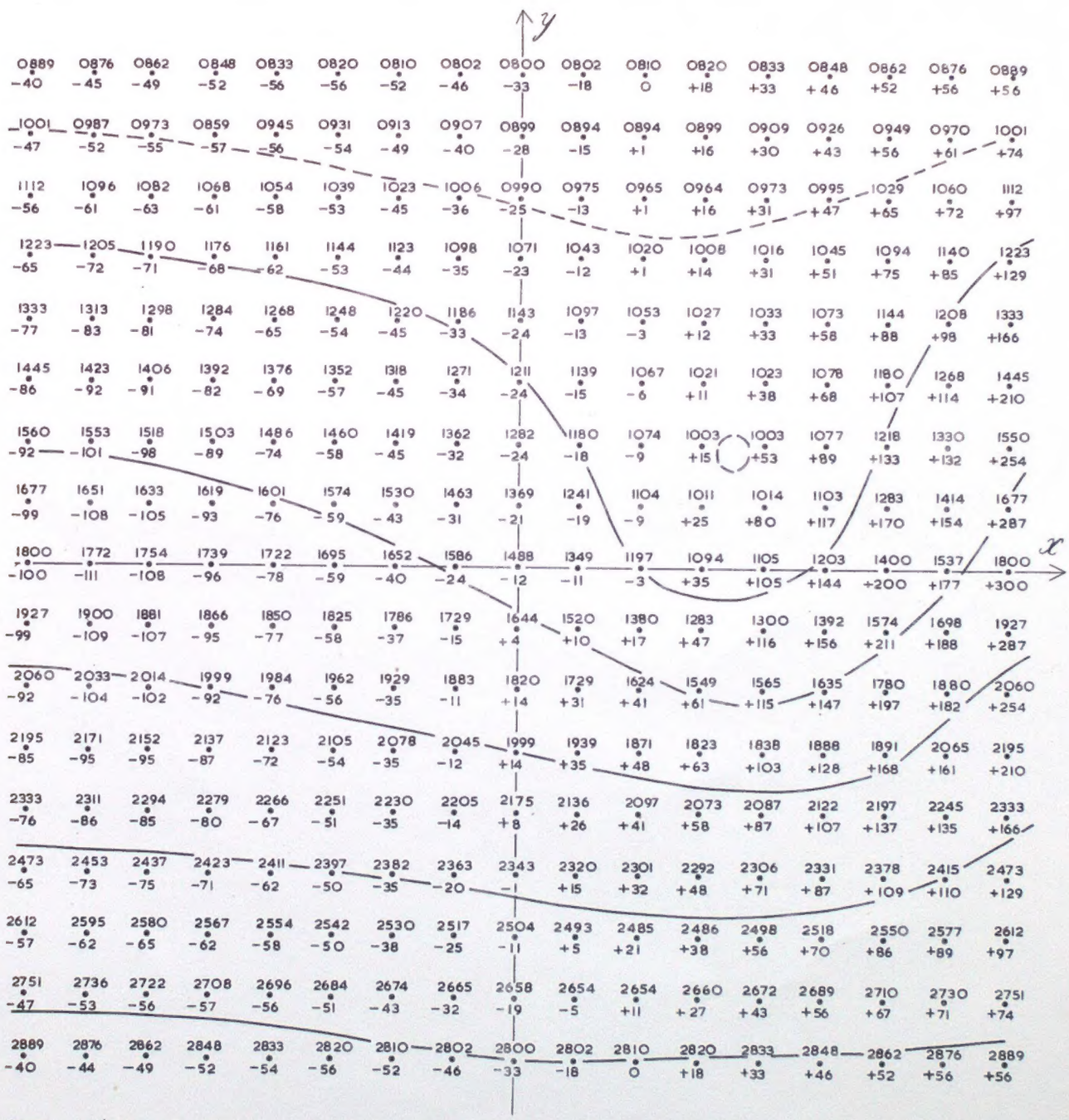

Fig. 6 (b) Final ψ field — Eulerian integration with fixed boundary values a = 4 b = 8 (16 time steps)